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The virtual mass of a closed torus in axisymmetric motion

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SUMMARY

An exact inviscid solution for the virtual-mass coefficient of a closed torus in axisymmetric motion is presented. It is also shown that the virtual-mass coefficient k for an open torus is bounded by $1.0 \leq k \leq 1.0281$.

1. Introduction

There is currently a growing interest in the hydrodynamics of toriodal shapes for vortex-ring models. Toroidal shapes are also frequently encountered in plasma physics, bio-physics and hydrodynamics of superfluids. In the hydrodynamical analysis of vortex rings and turbulent thermals, an important parameter which has to be considered is the virtual-mass coefficient. The effect of the virtual mass was found to be of primary significance in analysing the motion of a turbulent thermal in an unstratified environment [1]. In general, there exists three distinct virtual-inertia coefficients for a torus: two added-mass coefficients for motions along the symmetry axis and along the transverse axis, and one added-inertia coefficient for rotation about the transverse axis. These hydrodynamical coefficients may be determined from the energy of the fluid under the assumption of inviscid and incompressible flow.

An exact solution for the three added-inertia coefficients of an open torus, was recently published [2] by employing a toroidal coordinate system. Open and closed tori are distinguished by considering the ratio between the core radius a and the radius of the ring b: for an open torus $a/b < 1$ whereas for closed torus $a/b = 1$. It should be noted that the closed-torus solution can not be obtained as a limiting case of the open-torus solution, since for $a/b \approx 1$ the series solution becomes more slowly convergent as the size of the hole shrinks to zero.

The closed torus is a shape of particular interest both as a limiting case of the open torus and as a fundamental shape in modelling biophysical flows. It is customary in bio-fluid mechanics studies to approximate the roulade shape of red blood cells by a circular disk. Dorrepaal et al [3] studied the Stokes flow about a closed torus and suggested that a better model for the red blood cells be a closed torus rather than a circular disk. It was also shown in the same article that the flow properties associated with the closed torus in a steady Stokes flow are vastly different from those for a circular disk. A comparison between the viscous drag and torque on these two shapes also shows a considerable difference. In addition to the viscous effects, the inertia effects also play an important role since the flow in the cardio-vascular system is periodic

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(pulsating). For these reasons, the virtual mass of a closed torus moving along its axis of symmetry should be determined. The present paper provides an exact inviscid solution for this added-mass coefficient by using the tangent-sphere coordinate system which is particularly tailored for the closed torus.

2. Mathematical analysis

Following Moon and Spencer [4], an orthogonal tangent-sphere coordinate system (μ, ν, ψ) is defined in terms of the Cartesian coordinates (x, y, z) by the following transformation

$$x = \frac{\mu \cos \psi}{\mu^2 + \nu^2}; \qquad y = \frac{\mu \sin \psi}{\mu^2 + \nu^2}; \qquad z = \frac{\nu}{\mu^2 + \nu^2} \tag{1}$$

where $\infty > \mu > 0$, $\infty > \nu > -\infty$ and $2\pi > \psi \geqslant 0$. The coordinate surface given by eq. (1) may be also written as

$$x^2 + y^2 + z^2 = \frac{1}{\mu}\sqrt{x^2 + y^2} \tag{2}$$

implying that the surface $\mu = \text{const}$ is a closed torus with the symmetry axis in the z-direction. The surface $\mu = 1/(2r)$, for example, is a closed torus with core diameter equal to $2r$. Hence, at an infinite distance from the torus $\mu \to 0$, while at the origin $\mu \to \infty$.

A normal axisymmetric (independent of ψ) solution of the Laplace equation in tangent-sphere coordinates is

$$\phi(\mu, \nu) = (\mu^2 + \nu^2)^{\frac{1}{2}} \int_{-\infty}^{\infty} A(s) I_0(s\mu) \exp(-is\nu) ds \tag{3}$$

where I_m denotes the modified Bessel function of the first kind and of order m. Next consider a unit-velocity uniform axisymmetric flow about a torus given by $\mu = 1$. The Neumann boundary condition

$$\frac{\partial \phi(\mu, \nu)}{\partial \mu} = \frac{\partial z}{\partial \mu}, \qquad \mu = 1 \tag{4}$$

yields the following equation for the complex function $A(s)$:

$$\frac{2\nu}{(1 + \nu^2)^{\frac{3}{2}}} = - \int_{-\infty}^{\infty} \exp(-is\nu) \left\{ A(s) [I_0(s) + sI_1(s)] - \frac{d^2}{ds^2} [sI_1(s) A(s)] \right\} ds$$
$$= \int_{-\infty}^{\infty} \frac{1}{sI_1(s)} \frac{d}{ds} \left[s^2 I_1^2(s) \frac{dA(s)}{ds} \right] \exp(-is\nu) ds \tag{5}$$

where in obtaining the above the Riemann-Lebesgue lemma and two integrations by parts have been used. Inversion of the Fourier transform given in (5) yields

$$\frac{1}{sI_1(s)}\frac{d}{ds}\,[s^2 I_1^2(s)\,\frac{dA(s)}{ds}] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\nu\exp(-is\nu)}{(1+\nu^2)^{\frac{3}{2}}}\,d\nu$$

$$= \frac{2is}{\pi}\int_0^{\infty}\frac{\cos(s\nu)}{(1+\nu^2)^{\frac{1}{2}}}\,d\nu = \frac{2i}{\pi}s\,K_0(s) \qquad (6)$$

where K_m is the modified Bessel function of the second kind of order m. Integration of equation (6) with the proper asymptotic behaviour of $A(s)$ and its first derivative yields

$$\frac{dA(s)}{ds} = \frac{is}{2\pi I_1^2(s)}[I_1(s)K_0(s) + I_2(s)K_1(s)]\,. \qquad (7)$$

The virtual mass of the torus in axisymmetric motion is given by [2]

$$\lambda = -2\pi\rho\int_{-\infty}^{\infty}\phi(\mu,\nu)\,\frac{\partial z}{\partial\mu}\,\frac{h_\nu h_\psi}{h_\mu}\,d\nu, \quad \mu = 1 \qquad (8)$$

where ρ is the fluid density and the metric coefficients of the transformation given in (1) are

$$h_\nu = h_\mu = \frac{1}{\mu^2+\nu^2}, \qquad h_\psi = \frac{\mu}{\mu^2+\nu^2}\,. \qquad (9)$$

Substitution of (1), (3) and (9) into (8) yields

$$\lambda = 4\pi\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(s)\exp(-is\nu)I_0(s)(1+\nu^2)^{-5/2}\nu\,d\nu\,ds. \qquad (10)$$

Integrating (10) by parts we get

$$\lambda = \frac{4\pi\rho i}{3}\int_0^{\infty}\int_{-\infty}^{\infty}A(s)I_0(s)\exp(-is\nu)(1+\nu^2)^{-3/2}s\,d\nu\,ds \qquad (11)$$

$$= \frac{8i\pi\rho}{3}\int_{-\infty}^{\infty}A(s)I_0(s)K_1(s)s^2\,ds.$$

By making use of the following relation,

$$\int_0^s s^2 I_0(s)K_1(s)ds = \frac{1}{4}\,s^3[I_0(s)K_1(s) + I_1(s)K_2(s)], \qquad (12)$$

equation (11) is integrated again by parts to give

$$\lambda = \frac{-2i\pi\rho}{3} \int_{-\infty}^{\infty} \frac{dA(s)}{ds} [I_0(s)K_1(s) + I_1(s)K_2(s)]s^3 ds. \tag{13}$$

Substituting (7) into (13) and using the Wronskian

$$I_m(s) K_{m+1}(s) + I_{m+1}(s)K_m(s) = s^{-1} \tag{14}$$

we obtain the following expression for the virtual-mass coefficient:

$$k = \frac{\lambda}{\rho \forall_T} = \frac{8}{3\pi^2} \int_0^{\infty} [\frac{s^2}{I_1^2(s)} - 4s^2 K_1^2(s)] ds. \tag{15}$$

Here $\forall_T = \pi^2/4$ is the volume of the torus $\mu = 1$ and the virtual-mass coefficient denotes the ratio between the virtual mass and the fluid mass displaced by the torus.

To evaluate the second integral in the right hand side of (15) we refer to the Nicholson integral expression [5] for the product of two modified Bessel functions,

$$K_m(s)K_n(s) = 2 \int_0^{\infty} K_{m+n}(2s\cosh t)\cosh[t(m-n)]\, dt$$

$$= 2 \int_0^{\infty} K_{m-n}(2s\cosh t)\cosh[t(m+n)]\, dt, \tag{16}$$

and to the following integral [6]:

$$\int_0^{\infty} t^m K_n(ct) dt = 2^{m-1} c^{-m-1} \Gamma(\frac{1+m+n}{2}) \Gamma(\frac{1+m-n}{2}) \tag{17}$$

where Γ denotes the Gamma function. Using (16) and (17) it can be shown that

$$\int_0^{\infty} s^2 K_1^2(s) ds = \frac{1}{4}\, \Gamma(\frac{1}{2}) \Gamma(\frac{5}{2}) \int_0^{\infty} \frac{dt}{(\cosh t)^3} = \Gamma(\frac{1}{2}) \Gamma(\frac{5}{2})\, B(\frac{3}{2}, \frac{3}{2}). \tag{18}$$

Here $B(x, y)$ denotes the Beta function given by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{19}$$

Substituting (19) in (18) with the proper values for the Gamma function we get

$$\int_0^{\infty} s^2 K_1^2(s) ds = \frac{3}{32} \pi^2. \tag{20}$$

The first integral on the r.h.s. of (15) was first introduced by Watson [7] in his studies connected with cylindrical wind tunnels. The same integral has also been computed numerically by

Smythe [8] using Weddle's integration rule with intervals of 0.1. The result which is correct within eight significant figures is

$$\int_0^\infty \frac{s^2}{I_1^2(s)} ds = 7.5060642. \qquad (21)$$

Finally, substituting (20) and (21) into (15) we find for the virtual-mass coefficient

$$k = 1.02806216. \qquad (22)$$

3. Summary and conclusions

The closed torus may be considered as a limiting case of an open torus where the core radius is equal to the ring radius. Still another interesting limiting case of the open torus is the slender torus for which the ratio between the core radius and the radius of the ring is zero. The flow about the slender torus is in fact two-dimensional and approximated by the flow about two circles an infinite distance apart. Clearly, the virtual-mass coefficient for the slender torus is therefore equal to one, as also found by Wu and Yates [9]. From the numerical solution given by Miloh et al [2] the added-mass coefficients of an open torus were found to be monotonic functions bounded below by the corresponding value for a slender torus, and above by the value of a closed torus. By applying a proper extrapolation to the limit $a/b \to 1$, it has been suggested [2] that $1.0 \leqslant k < 1.0625$. The main result of this paper is the determination of the upper bound of the virtual-mass coefficient, namely $k \leqslant 1.0281$. Hence for most practical purposes the added-mass coefficient of an open torus in axisymmetrical motion may be taken as $k = 1$. It is also instructive to compare this with the added-mass coefficients of two contiguous equal spheres moving in the direction normal to the line of centers. It has been found [10] that for such a geometry the added-mass coefficient is bounded by $0.5 \leqslant k \leqslant 0.6210$, where clearly the lower bound corresponds to the case where the two spheres are an infinite distance apart and the upper bound to the case of two touching spheres.

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